

The characteristic quasi-polynomials of the arrangements of root systems and mid-hyperplane arrangements

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Abstract

Let q be a positive integer. In [8], we proved that the cardinality of the complement of an integral arrangement, after the modulo q reduction, is a quasi-polynomial of q , which we call the characteristic quasi-polynomial. In this paper, we study general properties of the characteristic quasi-polynomial as well as discuss two important examples: the arrangements of reflecting hyperplanes arising from irreducible root systems and the mid-hyperplane arrangements. In the root system case, we present a beautiful formula for the generating function of the characteristic quasi-polynomial which has been essentially obtained by Ch. Athanasiadis [2] and by A. Blass and B. Sagan [3]. On the other hand, it is hard to find the generating function of the characteristic quasi-polynomial in the mid-hyperplane arrangement case. We determine them when the dimension is less than six.

Key words: characteristic quasi-polynomial, elementary divisor, hyperplane arrangement, root system, mid-hyperplane arrangement.

1 Introduction

Let S be an arbitrary $m \times n$ integral matrix without zero columns. For each positive integer $q \in \mathbb{Z}_{>0}$, denote $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$ and $\mathbb{Z}_q^\times = \mathbb{Z}_q \setminus \{0\}$. Consider the set

$$M_q(S) := \{\mathbf{z} = (z_1, \dots, z_m) \in \mathbb{Z}_q^m : \mathbf{z}S \in (\mathbb{Z}_q^\times)^n\},$$

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and its cardinality $|M_q(S)|$. In our recent paper [8], we showed that there exists a monic **quasi-polynomial** (periodic polynomial) $\chi_S(q)$ with integral coefficients of degree m such that

$$\chi_S(q) = |M_q(S)|, \quad q \in \mathbb{Z}_{>0}.$$

Note that the set $M_q(S)$ is the complement of an arrangement of hyperplanes in the following sense: Let S_1, S_2, \dots, S_n be the columns of S . Each set

$$H_{i,q} := \{\mathbf{z} = (z_1, \dots, z_m) \in \mathbb{Z}_q^m : \mathbf{z}S_i = 0\}, \quad 1 \leq i \leq n,$$

can be called a “hyperplane” in \mathbb{Z}_q^m by a slight abuse of terminology. Then

$$M_q(S) = \mathbb{Z}_q^m \setminus \bigcup_{i=1}^n H_{i,q}.$$

For a sufficiently large prime number q , $\chi_S(q)$ is known [2] to be equal to the **characteristic polynomial** [9, Def. 2.52] of the real arrangement consisting of the following hyperplanes (ignoring possible repetitions):

$$H_{i,\mathbb{R}} := \{\mathbf{z} = (z_1, \dots, z_m) \in \mathbb{R}^m : \mathbf{z}S_i = 0\}, \quad 1 \leq i \leq n.$$

It is thus natural to call the quasi-polynomial $\chi_S(q)$ the **characteristic quasi-polynomial** of S as in [8]. Let us define its **generating function**

$$\Phi_S(t) := \sum_{q=1}^{\infty} \chi_S(q)t^q.$$

We understand that $M_1(S) = \emptyset$ for $q = 1$ and hence the summation is in effect for $q \geq 2$.

In this paper, we study the characteristic quasi-polynomial $\chi_S(q)$ or equivalently its generating function $\Phi_S(t)$. In Section 2, we discuss general properties of the characteristic quasi-polynomials and their generating functions. In the subsequent chapters, we deal with two kinds of specific arrangements defined over \mathbb{Z} : the arrangements of reflecting hyperplanes arising from irreducible root systems (Section 3) and the mid-hyperplane arrangements (Section 4). Let R be an irreducible root system of rank m and $n = |R|/2$. We assume that an $m \times n$ integral matrix $S = S(R) = [S_{ij}]$ satisfies

$$R_+ = \left\{ \sum_{i=1}^m S_{ij}\alpha_i : j = 1, \dots, n \right\},$$

where R_+ is a set of positive roots and $B(R) = \{\alpha_1, \alpha_2, \dots, \alpha_m\}$ is the set of **simple roots** associated with R_+ . In other words, S is a coefficient matrix of R_+ with respect to the basis $B(R)$. Define the characteristic quasi-polynomial $\chi_R(q) := \chi_S(q)$ and the generating function $\Phi_R(t) := \Phi_S(t)$ for each irreducible root system R . Then $\chi_R(q)$ and $\Phi_R(t) := \Phi_S(t)$ depend only upon R . In Section 3, we present a beautiful formula for the generating function $\Phi_R(t)$ for every irreducible root system R . This formula has

been essentially proved by Ch. Athanasiadis in [2] and A. Blass and B. Sagan in [3]. (See [6] also.) In Theorem 3.1, we will state the formula in our language and include a proof following [2, 3, 6] for completeness. In Section 4, we will give a formula for the generating function $\Phi_S(t)$ when S is equal to the coefficient matrix for the mid-hyperplane arrangement of dimension less than six.

We are aided by the computer package PARI/GP [10].

2 Results on the characteristic quasi-polynomial of an integral matrix

Let $\chi_S(t)$ be the characteristic quasi-polynomial of an $m \times n$ integral matrix S without zero columns. Fix a nonempty $J \subseteq [n] := \{1, 2, \dots, n\}$ and define an $m \times |J|$ matrix S_J consisting of the columns of S corresponding to the set J . Let $e_{J,1}, \dots, e_{J,\ell(J)} \in \mathbb{Z}_{>0}$ be the elementary divisors of S_J numbered so that $e_{J,1} | e_{J,2} | \dots | e_{J,\ell(J)}$, where $\ell(J) := \text{rank } S_J$. Write $e(J) := e_{J,\ell(J)}$, and define the **lcm period** $\rho_0(S)$ of S by

$$\begin{aligned} \rho_0 = \rho_0(S) &:= \text{lcm}\{e(J) : J \subseteq [n], J \neq \emptyset\} \\ &= \text{lcm}\{e(J) : J \subseteq [n], 1 \leq |J| \leq \min\{m, n\}\}. \end{aligned}$$

Then it is known ([8, Theorem 2.4]) that the lcm period ρ_0 is a period of $\chi_S(t)$.

It is further shown in [8] that the constituents of the quasi-polynomial $\chi_S(t)$ are the same for all q 's with the same value of $\gcd\{\rho_0, q\}$. Let d be a positive integer which divides ρ_0 , and define a monic polynomial $P_d(t) = P_{S,d}(t)$ with integral coefficients of degree m by

$$(1) \quad \chi_S(q) = P_d(q) \quad \text{for all } q \in d + \rho_0 \mathbb{Z}_{\geq 0}.$$

Put

$$e(J, d) := \prod_{j=1}^{\ell(J)} \gcd\{e_{J,j}, d\}.$$

Then the following formula was essentially proved in our previous paper [8].

Theorem 2.1. *For each $d \in \mathbb{Z}_{>0}$ with $d | \rho_0$, the polynomial $P_d(t)$ is given by*

$$P_d(t) = \sum_{J \subseteq [n]} (-1)^{|J|} e(J, d) t^{m-\ell(J)},$$

where for $J = \emptyset$, we understand that $\ell(\emptyset) = 0$ and that $e(\emptyset, d) = 1$.

Proof. Obtained from [8, (10)] and the inclusion-exclusion principle. □

Theorem 2.2 ([8] Theorem 2.5). *The polynomial*

$$P_1(t) = \sum_{J \subseteq [n]} (-1)^{|J|} t^{m-\ell(J)}$$

is equal to the ordinary characteristic polynomial [9, Def. 2.52] of the real arrangement consisting of the hyperplanes (ignoring possible repetitions) $H_{1,\mathbb{R}}, H_{2,\mathbb{R}}, \dots, H_{n,\mathbb{R}}$.

Corollary 2.3. *Suppose $d, d' \in \mathbb{Z}_{>0}$ both divide ρ_0 , and assume the following condition holds true for some positive integer s : $\gcd\{e(J), d\} = \gcd\{e(J), d'\}$ for all $J \subseteq [n]$ with $|J| \leq s$. Then*

$$\deg\{P_d(t) - P_{d'}(t)\} < m - s.$$

In particular, we have $\deg\{P_d(t) - P_1(t)\} < m - s$ if $\gcd\{e(J), d\} = 1$ for all $J \subseteq [n]$ with $|J| \leq s$.

Proof. We apply Theorems 2.1 and 2.2. It is enough to show $e(J, d) = e(J, d')$ for $J \subseteq [n]$ with $\ell(J) \leq s$. We can choose a subset $J' \subseteq J$ such that $\ell(J') = |J'| = \ell(J) \leq s$. Then $\gcd\{e(J'), d\} = \gcd\{e(J'), d'\}$. Since $e(J)|e(J')$ [8, Lemma 2.3], $\gcd\{e(J), d\} = \gcd\{e(J), d'\}$. This shows $e(J, d) = \prod_{j=1}^{\ell(J)} \gcd\{e_{J,j}, d\} = \prod_{j=1}^{\ell(J)} \gcd\{e_{J,j}, e(J), d\} = \prod_{j=1}^{\ell(J)} \gcd\{e_{J,j}, e(J), d'\} = \prod_{j=1}^{\ell(J)} \gcd\{e_{J,j}, d'\} = e(J, d')$. \square

Corollary 2.4. *Suppose that $d \in \mathbb{Z}_{>0}$ and $d' \in \mathbb{Z}_{>0}$ both divide ρ_0 and that $\gcd\{d, d'\} = 1$. In addition, we assume the following condition holds true for some positive integer s :*

$$(2) \quad \gcd\{e(J), d\} = 1 \quad \text{or} \quad \gcd\{e(J), d'\} = 1$$

for all $J \subseteq [n]$ with $|J| \leq s$. Then

$$\deg\{P_1(t) + P_{dd'}(t) - P_d(t) - P_{d'}(t)\} < m - s.$$

Proof. Suppose $J \subseteq [n]$ with $\ell(J) \leq s$. It is enough to show

$$1 + e(J, dd') - e(J, d) - e(J, d') = 0.$$

We can choose a subset $J' \subseteq J$ such that $\ell(J') = |J'| = \ell(J) \leq s$. Then either $\gcd\{e(J'), d\} = 1$ or $\gcd\{e(J'), d'\} = 1$ by (2). Since $e(J)|e(J')$,

$$\gcd\{e(J), d\} = 1 \quad \text{or} \quad \gcd\{e(J), d'\} = 1.$$

This shows that either $e(J, d) = 1$ or $e(J, d') = 1$. We finally have

$$\begin{aligned} 0 &= \{1 - e(J, d)\}\{1 - e(J, d')\} = 1 - e(J, d) - e(J, d') + e(J, d)e(J, d') \\ &= 1 - e(J, d) - e(J, d') + e(J, dd'). \end{aligned}$$

\square

Corollary 2.5. *Suppose that $d \in \mathbb{Z}_{>0}$ and $d' \in \mathbb{Z}_{>0}$ both divide ρ_0 and that $\gcd\{d, d'\} = 1$. If $e(J)$ are prime powers or one for all J , we have $P_{dd'}(t) = P_d(t) + P_{d'}(t) - P_1(t)$.*

Proof. Easily follows from Corollary 2.4. \square

For the rest of this section we discuss general properties of the generating functions $\Phi_S(t)$ of characteristic quasi-polynomials. Let $\omega = \exp(2\pi i/\rho_0)$ which is a primitive ρ_0 's root of unity. By (1)

$$\Phi_S(t) = \sum_{d=1}^{\rho_0} \Phi_{S,d}(t), \quad \Phi_{S,d}(t) = \sum_{s=0}^{\infty} P_d(d + \rho_0 s) t^{d+\rho_0 s}.$$

Note that $\Phi_S(\omega^k t) = \sum_{d=1}^{\rho_0} \omega^{kd} \Phi_{S,d}(t)$. Therefore from the orthogonality relations among powers of ω , i.e., by the Fourier inversion, $\Phi_{S,d}(t)$ for each d can be recovered from $\Phi_S(t)$ by

$$(3) \quad \Phi_{S,d}(t) = \frac{1}{\rho_0} \sum_{k=1}^{\rho_0} \omega^{-kd} \Phi_S(\omega^k t).$$

This relation will be used in Example 3.5 below.

Taking a common denominator, we can express $\Phi_S(t)$ as a rational function

$$\Phi_S(t) = \frac{Q(t)}{(1 - t^{\rho_0})^{m+1}}, \quad \deg Q < (m+1)\rho_0.$$

In the numerator $Q(t)$ the powers $t^{d+\rho_0 s}$, $s = 0, 1, \dots$, correspond to P_d . Therefore as in (3) for each d we can extract these powers as

$$(4) \quad \Phi_{S,d}(t) = \frac{Q_d(t)}{(1 - t^{\rho_0})^{m+1}}, \quad Q_d(t) = \frac{1}{\rho_0} \sum_{k=1}^{\rho_0} \omega^{-kd} Q(\omega^k t).$$

Let

$$P_d(q) = \sum_{k=0}^m c_{d,k} q^k \quad (c_{d,k} \in \mathbb{Z}).$$

Then

$$Q_d(t) = (1 - t^{\rho_0})^{m+1} \sum_{s=0}^{\infty} P_d(d + \rho_0 s) t^{d+\rho_0 s} = (1 - t^{\rho_0})^{m+1} \sum_{k=0}^m c_{d,k} \sum_{s=0}^{\infty} (d + \rho_0 s)^k t^{d+\rho_0 s}.$$

Define polynomials $q_{d,k}(t)$ by

$$(5) \quad \sum_{s=0}^{\infty} (d + \rho_0 s)^k t^{d+\rho_0 s} = \frac{q_{d,k}(t)}{(1 - t^{\rho_0})^{k+1}} \quad (d = 1, \dots, \rho_0).$$

Then we obtain

$$(6) \quad Q_d(t) = \sum_{k=0}^m (1 - t^{\rho_0})^{m-k} c_{d,k} q_{d,k}(t).$$

Now we present the following proposition, to which we give a proof because we were not able to find an appropriate reference in literature.

Proposition 2.6. *Define $q_{d,k}(t)$ by (5). Let $q_{d,k}^{(j)}(1)$ be their j -th derivatives at $t = 1$. Then*

$$(7) \quad 0 \neq q_{1,k}^{(j)}(1) = q_{2,k}^{(j)}(1) = \dots = q_{\rho_0,k}^{(j)}(1) \quad (j = 0, \dots, k).$$

Proof. For notational simplicity write $q_d(t) = q_{d,k}(t)$ and let

$$\tilde{q}_l(t) = \sum_{d=1}^{\rho_0} \omega^{ld} q_d(t) \quad (l = 1, \dots, \rho_0).$$

Then the inverse Fourier transform is

$$q_d(t) = \frac{1}{\rho_0} \sum_{l=1}^{\rho_0} \omega^{-ld} \tilde{q}_l(t) \quad (d = 1, \dots, \rho_0).$$

The j -th derivative of this at $t = 1$ is

$$q_d^{(j)}(1) = \frac{1}{\rho_0} \sum_{l=1}^{\rho_0} \omega^{-ld} \tilde{q}_l^{(j)}(1).$$

It follows that $q_d^{(j)}(1)$ does not depend on d if and only if

$$(8) \quad \tilde{q}_l^{(j)}(1) = 0 \quad (l = 1, \dots, \rho_0 - 1).$$

By the use of Eulerian numbers $W(k, h)$ (see Chapter III of [1]) we can write

$$(9) \quad \tilde{q}_l(t) = \left(\sum_{h=0}^{k-1} W(k, h) (\omega^l t)^{k-h} \right) (1 + \omega^l t + \omega^{2l} t^2 + \dots + \omega^{(\rho_0-1)l} t^{\rho_0-1})^{k+1}.$$

Note that $0 = 1 + \omega^l + \omega^{2l} + \dots + \omega^{(\rho_0-1)l}$ for $1 \leq l < \rho_0$. Therefore differentiating (9) with respect to t , we have $\tilde{q}_l^{(j)}(1) = 0$ for $1 \leq l < \rho_0$ and for $0 \leq j \leq k$. Thus $q_{1,k}^{(j)}(1) = q_{2,k}^{(j)}(1) = \dots = q_{\rho_0,k}^{(j)}(1)$. By summing up (5) we have

$$\sum_{d=1}^{\rho_0} q_{d,k}(t) = (1 - t^{\rho_0})^{k+1} \sum_{q=1}^{\infty} q^k t^q.$$

Since the Eulerian numbers are positive integers, it is not hard to see that the right hand side is a polynomial of degree $\geq k$ with positive integer coefficients. Thus $q_{d,k}^{(j)}(1)$ is not zero for $0 \leq j \leq k$, $1 \leq d \leq \rho_0$. \square

Proposition 2.6 and (6) imply that

$$P_d(t) = P_{d'}(t) \Leftrightarrow c_{d,k} = c_{d',k} \text{ for } 0 \leq k \leq m \Leftrightarrow Q_d^{(j)}(1) = Q_{d'}^{(j)}(1) \text{ for } 0 \leq j \leq m.$$

Furthermore note that lower order derivatives of Q_d at $t = 1$ determine coefficients of higher degree terms in $P_d(t)$. Therefore in terms of the generating function the relations in Corollaries 2.3 and 2.4 can be written as follows:

$$\deg\{P_d(t) - P_{d'}(t)\} < m - s \Leftrightarrow Q_d^{(j)}(1) = Q_{d'}^{(j)}(1) \quad (j = 0, 1, \dots, s),$$

$$\begin{aligned} \deg\{P_1(t) + P_{dd'}(t) - P_d(t) - P_{d'}(t)\} &< m - s \\ \Leftrightarrow Q_1^{(j)}(1) + Q_{dd'}^{(j)}(1) - Q_d^{(j)}(1) - Q_{d'}^{(j)}(1) &= 0 \quad (j = 0, 1, \dots, s). \end{aligned}$$

3 Arrangements of root systems

Let V be an m -dimensional Euclidean space and E be the affine space underlying V . Let R be an irreducible root system in V of rank m and $n = |R|/2$. Suppose that R_+ is a set of positive roots and $B = \{\alpha_1, \dots, \alpha_m\}$ is the set of simple roots associated with R_+ . Denote the coefficient matrix (with an arbitrary order of columns) of the positive roots R_+ with respect to B by $S = [S_{ij}]$, which is an $m \times n$ matrix:

$$R_+ = \left\{ \sum_{i=1}^m S_{ij} \alpha_i : j = 1, \dots, n \right\}.$$

In this section we give an explicit formula for the generating function $\Phi_R(t) := \Phi_S(t)$. The formula was essentially proved by Ch. Athanasiadis [2] and by A. Blass and B. Sagan [3]. (See [6] also.) Let

$$\tilde{\alpha} = \sum_{i=1}^m n_i \alpha_i$$

be the highest root. Then it is well-known that $h := 1 + \sum_{i=1}^m n_i$ is equal to the **Coxeter number**; see [4, Ch. VI, §1, 11. Prop. 31].

Theorem 3.1.

$$\Phi_R(t) = \frac{(n_1 \cdots n_m)(m!)t^h}{(1-t) \prod_{i=1}^m (1-t^{n_i})}.$$

Before proving this formula after [2, 3, 6], we introduce basic concepts. Let β_1, \dots, β_m be the basis for V which is dual to the basis B : $(\alpha_i, \beta_j) = \delta_{ij}$. Define a free abelian group

$$P(R^\vee) = \mathbb{Z}\beta_1 + \mathbb{Z}\beta_2 + \dots + \mathbb{Z}\beta_m$$

of rank m . Let

$$H_{i,k} = \{x \in E : (x, \alpha_i) = k\}$$

for $1 \leq i \leq n$ and $k \in \mathbb{Z}$. Then $\mathcal{A}_a = \{H_{i,k} : 1 \leq i \leq n, k \in \mathbb{Z}\}$ is an arrangement of (infinitely many) affine hyperplanes in E . The reflection with respect to $H_{i,k}$ is denoted by $s_{i,k}$:

$$s_{i,k}(x) = x - 2 \frac{(x, \alpha_i) - k}{(\alpha_i, \alpha_i)} \alpha_i \quad (x \in E).$$

The **affine Weyl group** W_a is the group generated by $\{s_{i,k} : 1 \leq i \leq n, k \in \mathbb{Z}\}$. Each connected component of $E \setminus \bigcup \mathcal{A}_a$ is called an **alcove**. The closure of an alcove is a fundamental domain of the group W_a acting on E [4, Ch. VI, §2, 1]. Consider a special alcove

$$\begin{aligned} C &= \{x \in E : 0 < (x, \alpha) < 1 \ (\alpha \in R_+)\} \\ &= \{x \in E : 0 < (x, \alpha_i) < 1 \ (i = 1, \dots, m), \ (x, \tilde{\alpha}) < 1\} \\ &= \left\{ \sum_{i=1}^m c_i \beta_i : 0 < c_i \ (i = 1, \dots, m), \ \sum_{i=1}^m c_i n_i < 1 \right\} \end{aligned}$$

which is an open simplex with vertices 0 and the β_i/n_i ($1 \leq i \leq m$).

Proof of Theorem 3.1. ([2, 3, 6]) Let A be the parallelepiped defined by

$$A = \left\{ \sum_{i=1}^m c_i \beta_i : 0 < c_i < 1 \ (i = 1, \dots, m) \right\}.$$

Then A is, by definition, a union of alcoves. The number of alcoves in A is equal to

$$\frac{\text{vol}(A)}{\text{vol}(C)} = (n_1 \cdots n_m)(m!);$$

see [4, Ch. VI, §2, 4. (5)]. Compute

$$\begin{aligned} \chi_R(t) &= |M_q(S)| = |\{\mathbf{z} = (z_1, \dots, z_m) \in \mathbb{Z}_q^m : \mathbf{z}S \in (\mathbb{Z}_q^\times)^n\}| \\ &= |\{\gamma \in A \cap \frac{1}{q}P(R^\vee) : \gamma \notin \bigcup \mathcal{A}_a\}| \\ &= |(A \cap \frac{1}{q}P(R^\vee)) \setminus \bigcup \mathcal{A}_a| \\ &= (n_1 \cdots n_m)(m!) |C \cap \frac{1}{q}P(R^\vee)|. \end{aligned}$$

On the other hand,

$$\begin{aligned}
|C \cap \frac{1}{q}P(R^\vee)| &= |\{\sum_{i=1}^m c_i \beta_i : c_i \in \{\frac{1}{q}, \frac{2}{q}, \dots, \frac{q-1}{q}\} \ (i = 1, \dots, m), \ \sum_{i=1}^m c_i n_i < 1\}| \\
&= |\{\sum_{i=1}^m c_i \beta_i : c_i \in \mathbb{Z}_{>0} \ (i = 1, \dots, m), \ \sum_{i=1}^m c_i n_i < q\}|.
\end{aligned}$$

This function is known as the Ehrhart quasi-polynomial of the open simplex bounded by the coordinate hyperplanes and the hyperplane $\sum_{i=1}^m c_i x_i = q$; see [11, page 235ff]. Thus

$$\begin{aligned}
\Phi_R(t) &= (n_1 \cdots n_m)(m!) \sum_{q=1}^{\infty} |C \cap \frac{1}{q}P(R^\vee)| t^q \\
&= \frac{(n_1 \cdots n_m)(m!) t^{n_1 + \cdots + n_m + 1}}{(1-t) \prod_{i=1}^m (1-t^{n_i})} = \frac{(n_1 \cdots n_m)(m!) t^h}{(1-t) \prod_{i=1}^m (1-t^{n_i})}.
\end{aligned}$$

This completes the proof. \square

Corollary 3.2. *The minimum period of the characteristic quasi-polynomial for an irreducible root system is equal to $\text{lcm}(n_1, \dots, n_m)$.*

Proof. The assertion holds true because by Proposition 4.4.1 of [11] the minimum period of the Ehrhart quasi-polynomial of the open simplex bounded by the coordinate hyperplanes and the hyperplane $\sum_{i=1}^m n_i x_i = q$ is equal to $\text{lcm}(n_1, \dots, n_m)$. \square

The minimum periods for all irreducible root systems are shown in the following table:

root system	n_1, n_2, \dots, n_m	$h = 1 + n_1 + n_2 + \cdots + n_m$	minimum period
A_m	$1, 1, \dots, 1, 1$	$m + 1$	1
B_m	$1, 2, 2, \dots, 2$	$2m$	2
C_m	$2, 2, \dots, 2, 1$	$2m$	2
D_m	$1, 2, 2, \dots, 2, 1, 1$	$2m - 2$	2
E_6	$1, 2, 2, 3, 2, 1$	12	6
E_7	$2, 2, 3, 4, 3, 2, 1$	18	12
E_8	$2, 3, 4, 6, 5, 4, 3, 2$	30	60
F_4	$2, 3, 4, 2$	12	12
G_2	$2, 3$	6	6

Remark 3.3. *With PARI/GP we checked that for every irreducible root system the minimum period coincides with the lcm period $\rho_0(S)$.*

Corollary 3.4. *Let q be a positive integer. For an irreducible root system R with its Coxeter number h , $\chi_R(q) > 0$ if and only if $q \geq h$.*

Proof. The lowest non-zero term of $\Phi_R(t)$ is equal to $(n_1 \cdots n_m)(m!)t^h$. \square

Example 3.5. (B_m, C_m, D_m) By Theorem 3.1 and the table above, we have the generating functions of type B_m, C_m, D_m :

$$\Phi_{B_m}(t) = \Phi_{C_m}(t) = \frac{2^{m-1}(m!)t^{2m}}{(1-t)^2(1-t^2)^{m-1}}, \quad \Phi_{D_m}(t) = \frac{2^{m-3}(m!)t^{2m-2}}{(1-t)^4(1-t^2)^{m-3}}.$$

Thus the characteristic quasi-polynomials of B_m is the same as the characteristic quasi-polynomials of C_m . Since the minimum periods of these three root systems are all equal to two, there exist four polynomials $P_1(q), P_2(q), Q_1(q), Q_2(q)$ satisfying

$$\Phi_1(t) := \Phi_{B_m}(t) = \Phi_{C_m}(t) = \sum_{i=0}^{\infty} P_1(2i+1)t^{2i+1} + \sum_{i=1}^{\infty} P_2(2i)t^{2i},$$

$$\Phi_2(t) := \Phi_{D_m}(t) = \sum_{i=0}^{\infty} Q_1(2i+1)t^{2i+1} + \sum_{i=1}^{\infty} Q_2(2i)t^{2i}.$$

Compute

$$\begin{aligned} 2 \sum_{i=1}^{\infty} P_2(2i)t^{2i} &= \Phi_1(t) + \Phi_1(-t) = \frac{2^{m-1}(m!)t^{2m}}{(1-t)^2(1-t^2)^{m-1}} + \frac{2^{m-1}(m!)t^{2m}}{(1+t)^2(1-t^2)^{m-1}} \\ &= \frac{2^m(m!)t^{2m}(1+t^2)}{(1-t^2)^{m+1}} \end{aligned}$$

and

$$\begin{aligned} 2 \sum_{i=1}^{\infty} Q_1(2i+1)t^{2i+1} &= \Phi_2(t) - \Phi_2(-t) = \frac{2^{m-3}(m!)t^{2m-2}}{(1-t)^4(1-t^2)^{m-3}} - \frac{2^{m-3}(m!)t^{2m-2}}{(1+t)^4(1-t^2)^{m-3}} \\ &= \frac{2^m(m!)t^{2m+1}(1+t^2)}{(1-t^2)^{m+1}} = 2 \sum_{i=1}^{\infty} P_2(2i)t^{2i+1}. \end{aligned}$$

This implies

$$P_2(q) = Q_1(q-1).$$

Since $Q_1(q)$ is equal to the ordinary characteristic polynomial of D_m by Theorem 2.2, we obtain

$$P_2(q) = Q_1(q-1) = (q-2)(q-4) \cdots (q-2m+2)(q-m).$$

Actually we may derive the following characteristic quasi-polynomials from the generating functions $\Phi_1(t)$ and $\Phi_2(t)$:

$$\begin{aligned} \chi_{B_m}(q) &= \chi_{C_m}(q) = \begin{cases} (q-1)(q-3) \cdots (q-2m+1) & \text{if } q \text{ is odd,} \\ (q-2)(q-4) \cdots (q-2m+2)(q-m) & \text{if } q \text{ is even,} \end{cases} \\ \chi_{D_m}(q) &= \begin{cases} (q-1)(q-3) \cdots (q-2m+3)(q-m+1) & \text{if } q \text{ is odd,} \\ (q-2)(q-4) \cdots (q-2m+4) \left\{ q^2 - 2(m-1)q + \frac{m(m-1)}{2} \right\} & \text{if } q \text{ is even.} \end{cases} \end{aligned}$$

Remark 3.6. We may also prove $\chi_{B_m}(2q) = \chi_{D_m}(2q-1)$ by constructing a one-to-one correspondence between $M_{2q}(S(B_m))$ and $M_{2q-1}(S(D_m))$.

Example 3.7. Let R be a root system of type E_6 . We use PLATE V in [4] to get the 6×36 matrix $S = S(E_6)$:

$$S(E_6) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Thus $(n_1, n_2, n_3, n_4, n_5, n_6) = (1, 2, 2, 3, 2, 1)$ and $h = 12$. By Theorem 3.1, we have the generating function of type E_6 :

$$\Phi_{E_6}(t) = \frac{24 \cdot (6!)t^{12}}{(1-t)^3(1-t^2)^3(1-t^3)}.$$

By expanding this formal power series we have the characteristic quasi-polynomial of E_6 :

$$\chi_{E_6}(q) = \begin{cases} q^6 - 36q^5 + 510q^4 - 3600q^3 + 13089q^2 - 22284q + 12320 \\ \quad = (q-1)(q-4)(q-5)(q-7)(q-8)(q-11), \\ \quad \gcd\{6, q\} = 1, \\ q^6 - 36q^5 + 510q^4 - 3600q^3 + 13224q^2 - 23904q + 16640 \\ \quad = (q-2)(q-4)(q-8)(q-10)(q^2 - 12q + 26), \\ \quad \gcd\{6, q\} = 2, \\ q^6 - 36q^5 + 510q^4 - 3600q^3 + 13089q^2 - 22284q + 12960 \\ \quad = (q-3)(q-9)(q^4 - 24q^3 + 195q^2 - 612q + 480), \\ \quad \gcd\{6, q\} = 3, \\ q^6 - 36q^5 + 510q^4 - 3600q^3 + 13224q^2 - 23904q + 17280 \\ \quad = (q-6)^2(q^4 - 24q^3 + 186q^2 - 504q + 480), \\ \quad \gcd\{6, q\} = 6. \end{cases}$$

We have computed $\{e(J) : |J| \leq 1\} = \{e(J) : |J| \leq 2\} = \{e(J) : |J| \leq 3\} = \{1\}$, $\{e(J) : |J| \leq 4\} = \{e(J) : |J| \leq 5\} = \{1, 2\}$, $\{e(J) : |J| \leq 6\} = \{1, 2, 3\}$ and the constituents of the quasi-polynomial are consistent with Corollaries 2.3 and 2.4.

Remark. R. Suter [12] gave essentially the same calculation for every irreducible root system.

4 Mid-hyperplane arrangement

The mid-hyperplane arrangement was defined in [7] to find the number of “ranking patterns” generated by the unidimensional unfolding model in mathematical psychology ([5]). This arrangement is defined as follows.

Let $m \geq 4$ be an integer. We define two kinds of hyperplanes as follows:

$$\begin{aligned} H_{ij} &:= \{(\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m : \alpha_i = \alpha_j\}, \quad 1 \leq i < j \leq m, \\ H_{ijkl} &:= \{(\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m : \alpha_i + \alpha_j = \alpha_k + \alpha_l\}, \quad (i, j, k, l) \in I_4, \end{aligned}$$

where $I_4 := \{(i, j, k, l) : 1 \leq i < j \leq m, i < k < l \leq m, j \text{ is different from } k \text{ and } l\}$. Then the mid-hyperplane arrangement \mathcal{M}_m is defined as

$$\mathcal{M}_m := \{H_{ij} \ (1 \leq i < j \leq m), H_{ijkl} \ ((i, j, k, l) \in I_4)\}.$$

Now, let $T(\mathcal{M}_m) : m \times n$ be the coefficient matrix of \mathcal{M}_m , where $n = |\mathcal{M}_m| = \binom{m}{2} + 3\binom{m}{4}$.

4.1 Characteristic quasi-polynomial and generating function of \mathcal{M}_4

When $m = 4$, we have $\{e(J) : |J| \leq 1\} = \{1\}$, $\{e(J) : |J| \leq 2\} = \{e(J) : |J| \leq 3\} = \{e(J) : |J| \leq 4\} = \{1, 2\}$, and thus $\rho_0 = 2$. The characteristic quasi-polynomial is

$$\chi_{T(\mathcal{M}_4)}(q) = \begin{cases} q^4 - 9q^3 + 23q^2 - 15q = q(q-1)(q-3)(q-5) & \text{if } q \text{ is odd,} \\ q^4 - 9q^3 + 26q^2 - 24q = q(q-2)(q-3)(q-4) & \text{if } q \text{ is even.} \end{cases}$$

From this characteristic quasi-polynomial, we obtain

$$\Phi_{T(\mathcal{M}_4)}(t) = \frac{48t^6(t^3 + 5t^2 + 7t + 3)}{(1 - t^2)^5} = \frac{48t^6(t + 3)}{(1 - t)^5(1 + t)^3}.$$

4.2 Characteristic quasi-polynomial and generating function of \mathcal{M}_5

When $m = 5$, we have $\{e(J) : |J| \leq 1\} = \{1\}$, $\{e(J) : |J| \leq 2\} = \{1, 2\}$, $\{e(J) : |J| \leq 3\} = \{1, 2, 3\}$, $\{e(J) : |J| \leq 4\} = \{e(J) : |J| \leq 5\} = \{1, 2, 3, 4, 5, 6\}$, and thus $\rho_0 = 60$. With the help of the relations

$$P_d(t) + P_{d'}(t) - P_{dd'}(t) - P_1(t) = \begin{cases} 0 & \text{for } (d, d') = (2, 5), (3, 5), (4, 5), (5, 6), (5, 12), \\ -120t & \text{for } (d, d') = (2, 3), (2, 15), (3, 4), (3, 10), (3, 20), (4, 15) \end{cases}$$

(see Corollary 2.4), we obtained the constituents of $\chi_{T(\mathcal{M}_5)}(q)$ as

$$P_1(q) = q^5 - 25q^4 + 215q^3 - 695q^2 + 504q$$

$$\begin{aligned}
&= q(q-1)(q-7)(q-8)(q-9), \\
P_2(q) &= q^5 - 25q^4 + 230q^3 - 920q^2 + 1104q \\
&= q(q-2)(q^3 - 23q^2 + 184q - 552), \\
P_3(q) &= q^5 - 25q^4 + 215q^3 - 735q^2 + 864q \\
&= q(q-3)(q-9)(q^2 - 13q + 32), \\
P_4(q) &= q^5 - 25q^4 + 230q^3 - 920q^2 + 1344q \\
&= q(q-4)(q-6)(q-7)(q-8), \\
P_5(q) &= q^5 - 25q^4 + 215q^3 - 695q^2 + 600q \\
&= q(q-5)(q^3 - 20q^2 + 115q - 120), \\
P_6(q) &= q^5 - 25q^4 + 230q^3 - 960q^2 + 1584q \\
&= q(q-6)(q^3 - 19q^2 + 116q - 264), \\
P_{10}(q) &= q^5 - 25q^4 + 230q^3 - 920q^2 + 1200q \\
&= q(q-10)(q^3 - 15q^2 + 80q - 120), \\
P_{12}(q) &= q^5 - 25q^4 + 230q^3 - 960q^2 + 1824q \\
&= q(q^4 - 25q^3 + 230q^2 - 960q + 1824), \\
P_{15}(q) &= q^5 - 25q^4 + 215q^3 - 735q^2 + 960q \\
&= q(q^4 - 25q^3 + 215q^2 - 735q + 960), \\
P_{20}(q) &= q^5 - 25q^4 + 230q^3 - 920q^2 + 1440q \\
&= q(q^4 - 25q^3 + 230q^2 - 920q + 1440), \\
P_{30}(q) &= q^5 - 25q^4 + 230q^3 - 960q^2 + 1680q \\
&= q(q^4 - 25q^3 + 230q^2 - 960q + 1680), \\
P_{60}(q) &= q^5 - 25q^4 + 230q^3 - 960q^2 + 1920q \\
&= q(q^4 - 25q^3 + 230q^2 - 960q + 1920).
\end{aligned}$$

The generationg function is

$$\begin{aligned}
\Phi_{T(\mathcal{M}_5)}(t) &= 240t^{11}(6t^{20} + 40t^{19} + 112t^{18} + 282t^{17} + 511t^{16} \\
&\quad + 917t^{15} + 1301t^{14} + 1818t^{13} + 2163t^{12} + 2493t^{11} \\
&\quad + 2479t^{10} + 2462t^9 + 2078t^8 + 1734t^7 + 1263t^6 \\
&\quad + 903t^5 + 523t^4 + 308t^3 + 137t^2 + 59t + 11) \\
&\quad / \{(1-t)^6(1+t)^4(1-t+t^2)^2(1+t+t^2)^3 \\
&\quad (1+t^2)^2(1+t+t^2+t^3+t^4)^2\}.
\end{aligned}$$

Remark 4.1. For \mathcal{M}_6 , we have $\{e(J) : |J| \leq 1\} = \{1\}$, $\{e(J) : |J| \leq 2\} = \{1, 2\}$, $\{e(J) : |J| \leq 3\} = \{1, 2, 3\}$, $\{e(J) : |J| \leq 4\} = \{1, 2, 3, 4, 5, 6\}$, $\{e(J) : |J| \leq 5\} = \{e(J) : |J| \leq 6\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$, and thus $\rho_0 = 27720$. It was computationally infeasible for us to obtain $\chi_{T(\mathcal{M}_6)}(q)$, so we cannot tell whether this ρ_0 is the minimum period or not.

References

- [1] M. Aigner, *Combinatorial Theory*, Springer, Berlin, 1979.
- [2] C. A. Athanasiadis, Characteristic polynomials of subspace arrangements and finite fields, *Adv. Math.* **122** (1996), 193–233.
- [3] A. Blass and B. Sagan, Characteristic and Ehrhart polynomials, *J. Algebraic Combin.* **7** (1998), 115–126.
- [4] N. Bourbaki, *Lie Groups and Lie Algebras: Chapters 4-6*, Springer-Verlag, Berlin-Heidelberg-New York, 2002.
- [5] C. H. Coombs, *A Theory of Data*, John Wiley & Sons, New York, 1964.
- [6] M. Haiman, Conjectures on the quotient ring of diagonal invariants, *J. Algebraic Combin.* **3** (1994), 17–76.
- [7] H. Kamiya, P. Orlik, A. Takemura and H. Terao, Arrangements and ranking patterns, *Ann. Comb.* **10** (2006), 219–235.
- [8] H. Kamiya, A. Takemura and H. Terao, Periodicity of hyperplane arrangements with integral coefficients modulo positive integers, to appear in *J. Alg. Combin.* doi:10.1007/s10801-007-0091-2.
- [9] P. Orlik and H. Terao, *Arrangements of Hyperplanes*, Springer-Verlag, Berlin, 1992.
- [10] PARI/GP (<http://pari.math.u-bordeaux.fr>).
- [11] R. Stanley, *Enumerative Combinatorics*, vol. I, Cambridge University Press, Cambridge, 1997.
- [12] R. Suter, The number of lattice points in alcoves and the exponents of the finite Weyl group, *Math. Comp.*, **67** (1998), 751–758.